# Analysis Qualifying Examination

Tuesday, January 19, 2021, Noon–5:00pm

**INSTRUCTIONS:** Work 5 of the following 6 problems. Write on only one side of each page. Each problem is worth 20 points.

**1)** Suppose: $(X, d)$ and $(Y, \rho)$ are metric spaces; $(Y, \rho)$ is compact; and $\phi : Y \to X$ is a continuous and onto function.

a) A well-known theorem states that if $F \subseteq Y$ is compact, then $\phi(F)$ is also compact. Prove this theorem, and conclude that $(X, d)$ is a compact metric space.

b) Suppose $G \subseteq X$ and $\phi^{-1}(G)$ is an open set. Prove that $G$ is an open set.

**2)** Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Being sure to include all details, prove that

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

(*Suggestion:* Notice that for $n, N \in \mathbb{N}$ and $n > N$, $\dfrac{a_1 + \cdots + a_N}{n} + \dfrac{a_{N+1} + \cdots + a_n}{n} = \dfrac{a_1 + \cdots + a_n}{n}$. Approximate $\dfrac{a_{N+1} + \cdots + a_n}{n}$ in terms of $\liminf\limits_{n\to\infty} a_n$ and examine what happens if you hold $N$ fixed and let $n$ grow.)

**3)** Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $\displaystyle\int_a^b |f(t)|\, dt = 0$.

a) If $f$ is continuous, prove that $f(t) = 0$ for every $t \in [a, b]$.

b) Give an example (with proof) of a non-zero Riemann integrable function such that $\int_a^b |f(t)|\, dt = 0$.

**4)** Let $Y \subseteq [0, 1]$ be the usual Cantor set[1] and let $C(Y)$ be the collection of all continuous complex-valued functions on $Y$. A function $p \in C(Y)$ is a *projection* if for every $y \in Y$, $p(y)^2 = p(y)$.

Given $f \in C(Y)$ and $\varepsilon > 0$, prove that there exists $n \in \mathbb{N}$, projections $\{p_k\}_{k=1}^{n} \subseteq C(Y)$ and complex numbers $\{\alpha_k\}_{k=1}^{n}$ such that for every $y \in Y$,

$$\left| f(y) - \sum_{k=1}^{n} \alpha_k p_k(y) \right| < \varepsilon.$$

**5)** Suppose $(a_n)$ is a decreasing sequence of real numbers and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} n a_n = 0$.

**6)** For $x \in \mathbb{R}$, consider the series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$.

a) Prove this series converges for every $x \in \mathbb{R}$.

b) Set $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$. Prove that $f$ is differentiable at each $x \in \mathbb{R}$. Also, find a formula for $f'(x)$ (in terms of a series), being sure to justify that your formula is correct.

[1] Recall this is the set obtained by removing $(1/3, 2/3)$ from $[0, 1]$, then removing the middle third from the remaining intervals, etc.